Priprema za predavanje

$\left\{ \begin{array}{c} \text{Razredbeni Ispiti} \\ \text{GIMNAZIJA} \\ \text{Rješenja} \end{array} \right\}$

Crveni tim

25. ožujka 2006.

$\mathcal{RJ}$ **1**

$$z = \frac{x + i\sqrt{2}}{1 - i} \cdot \frac{1 + i}{1 + i} = \frac{x + i\sqrt{2} + xi + i^2\sqrt{2}}{1 - (-1)} = \frac{x - \sqrt{2}}{2} + \frac{x - \sqrt{2}}{2} i$$

$$|z| = \sqrt{\left(\frac{x - \sqrt{2}}{2}\right)^2 + \left(\frac{x + \sqrt{2}}{2}\right)} = \sqrt{\frac{2x^2 + 4}{4}}$$

$$\frac{\sqrt{x^2 + 2}}{\sqrt{2}} = \sqrt{2}$$

$$\sqrt{x^2 + 2} = 2$$

$$x = \pm\sqrt{2}$$

Pa je naše rješenje $x = \sqrt{2}$.

$\mathcal{RJ}$ **2** *Uzmimo da je $z = a + bi$*

$$z + \frac{1}{z} = 1$$

$$z^2 + 1 = z$$

$$(a + bi)^2 + 1 = a + bi$$

$$a^2 - b^2 + 1 + 2abi = a + bi$$

$$\Rightarrow \begin{cases} a^2 - b^2 + 1 = a \\ 2ab = b \end{cases}$$

$$a = \frac{1}{2} \ i \ b^2 = \frac{3}{4}$$

$$|z| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$\mathcal{RJ}$ **3** *A kako da ja znam? :(Mrzim kompleksne brojeve!*

$\mathcal{RJ}$ **4** *Znamo da vrijedi* $z \cdot \overline{z} = a^2 + b^2$.

$$z = \frac{\sqrt{2} - i}{1 - i} \cdot \frac{1 + i}{1 + i} = \frac{\sqrt{2} - 1 + i(\sqrt{2} - 1)}{1 - (-1)} = \frac{\sqrt{2} - 1}{2} + \frac{\sqrt{2} - 1}{2} i$$

$$z \cdot \overline{z} = \left(\frac{\sqrt{2} - 1}{2} \right)^2 + \left(\frac{\sqrt{2} - 1}{2} \right)^2 = \frac{3}{2} - \sqrt{2}$$

$\mathcal{RJ}$ **5** *Samo raspišemo izraz iz zadatka...*

$$\left| \frac{1 - zi}{z} \right| \leq 1$$

$$\frac{|1 - zi|}{|z|} \leq 1$$

$$|1 - zi| \leq |z|$$

$$|1 - (a + bi)i| \leq |a + bi|$$

$$|1 + b - ai| \leq |a + bi|$$

$$\sqrt{(1 + b)^2 + (-a)^2} \leq \sqrt{a^2 + b^2}$$

$$1 + 2b \leq 0$$

$$b \leq -\frac{1}{2}$$

A upravo je $b = \operatorname{Im} z$. *Stoga možemo slobodno zaključiti da nejednakost pod* ***A*** *slijedi iz uvjeta u zadatku.*

$\mathcal{RJ}$ **6**

$$\frac{12}{3 + \sqrt{3}} \cdot \frac{3 - \sqrt{3}}{3 - \sqrt{3}} = \frac{12(3 - \sqrt{3})}{12} = 3 - \sqrt{3}$$

Znamo da je $|z| = r = \sqrt{a^2 + b^2}$. *Također vrijedi* $z^n = r^n(\cos\varphi + i \sin\varphi)$. *Očito mora vrijediti* $|z^n| = r^n$.

$$r = \sqrt{9 + 3} = \sqrt{12}$$

$$|z^{2002}| = r^{2002} = \left(r^2\right)^{1001} = 12^{1001}$$

$\mathcal{RJ}$ **7**

$$\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)^5=$$

$$\left(\frac{\sqrt{2}}{2}\right)^5+\binom{5}{1}\left(\frac{\sqrt{2}}{2}\right)^5 i+\binom{5}{2}\left(\frac{\sqrt{2}}{2}\right)^5 i^2+\binom{5}{3}\left(\frac{\sqrt{2}}{2}\right)^5 i^3+\binom{5}{4}\left(\frac{\sqrt{2}}{2}\right)^5 i^4+\left(\frac{\sqrt{2}}{2}\right)^5 i^5=$$

$$\left(\frac{\sqrt{2}}{2}\right)^5+5\left(\frac{\sqrt{2}}{2}\right)^5 i+10\left(\frac{\sqrt{2}}{2}\right)^5(-1)+10\left(\frac{\sqrt{2}}{2}\right)^5(-i)+5\left(\frac{\sqrt{2}}{2}\right)^5+\left(\frac{\sqrt{2}}{2}\right)^5 i=$$

$$\left(\frac{\sqrt{2}}{2}\right)^5-10\left(\frac{\sqrt{2}}{2}\right)^5+5\left(\frac{\sqrt{2}}{2}\right)^5+i\left(5\left(\frac{\sqrt{2}}{2}\right)^5-10\left(\frac{\sqrt{2}}{2}\right)^5+\left(\frac{\sqrt{2}}{2}\right)^5\right)=$$

$$-4\frac{4\sqrt{2}}{32}+-4\frac{4\sqrt{2}}{32}i=\frac{-\sqrt{2}}{2}+\frac{-\sqrt{2}}{2}i$$

$\mathcal{RJ}$ **8** *Kod ovog zadatka moramo pisati samo članove kod koji će biti realni. Stoga kod primjene binomne formule pišemo samo pribrojnike u kojima je k neparan.*

$$\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=$$

$$\left(\frac{-1}{2}\right)^9+\binom{9}{2}\left(\frac{-1^7}{2}\right)\left(\frac{\sqrt{3}}{2}\right)^2 i^2+\binom{9}{4}\left(\frac{-1^5}{2}\right)\left(\frac{\sqrt{3}}{2}\right)^4 i^4+$$

$$\binom{9}{6}\left(\frac{-1^3}{2}\right)\left(\frac{\sqrt{3}}{2}\right)6i^6+\binom{9}{8}\left(\frac{-1^1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)8i^8=$$

$$=\frac{257}{2^8}$$

$\mathcal{RJ}$ **9**

$$\frac{1}{i}+\frac{1}{i^2}+\frac{1}{i^3}+\frac{1}{i^4}=\frac{1}{i}+(-1)+\frac{-1}{i}+1=0.$$

$$\frac{1}{i}+\frac{1}{i^2}+\frac{1}{i^3}+\frac{1}{i^4}+\frac{1}{i^5}+\cdots+\frac{1}{i^100}=$$

$$\overbrace{\frac{1}{i^{4\cdot0+1}}+\frac{1}{i^{4\cdot0+2}}+\frac{1}{i^{4\cdot0+3}}+\frac{1}{i^{4\cdot0+4}}}+\overbrace{\frac{1}{i^{4\cdot1+1}}+\frac{1}{i^{4\cdot1+2}}+\frac{1}{i^{4\cdot1+3}}+\frac{1}{i^{4\cdot1+4}}}+\frac{1}{i^{4\cdot2+1}}+\cdots+\frac{1}{i^{4\cdot23+4}}+$$

$$+\overbrace{\frac{1}{i^{4\cdot24+1}}+\frac{1}{i^{4\cdot24+2}}+\frac{1}{i^{4\cdot24+3}}+\frac{1}{i^{4\cdot24+4}}}=25\cdot0=0$$

$\mathcal{RJ}$ **10**

$$(1+i)^n = (1-i)^n$$
$$(1+i)^2 = (1-i)^2$$
$$(2i) = -2i$$

Pokažimo da će za višekratnike broja 4 *jednakost biti ispunjena!*

$$(1+i)^{4m} = (1-i)^{4m}$$
$$\left((1+i)^2\right)^{2m} = \left((1-i)^2\right)^{2m}$$
$$\left((2i)^2\right)^m = \left((-2i)^2\right)^m$$

A lako je odrediti da postoji 51 *višekratnik broja* 4 *među brojevima manjim od* 2005.

Koristeći ovu činjenicu lako se dokazuje da nema drugih brojeva za koje bi gornja jednakost bila ispunjena. Naime, pretpostavimo da je jednakost ispunjena za neki broj koji nije višekratnik broja 4 *i pokažemo da to ne vrijedi.*

$\mathcal{RJ}$ **11** *Dva!*

$$ux^2 + v = 0$$
$$x^2 = \frac{-v}{u}$$
$$x = \pm\sqrt{\frac{v}{u}}$$